

Sleep as the solution to an optimization problem

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This paper develops a highly simplified model with which to analyze the phenomenon of sleep. Motivated by Crick's suggestion that sleep is the brain's way of "taking out the trash," a suggestion that is supported by emerging evidence, we consider the problem of the filling and emptying of a tank. At any given time, the tank may take in external resource, or fill, if resource is available at that time, or it may empty. The filling phases correspond to information input from the environment, or input of some material in general, while the emptying phases correspond to processing of the resource. Given a resource-availability profile over some time interval T , we develop a canonical algorithm for determining the fill-empty profile that produces the maximum quantity of processed resource at the end of the time interval. From this algorithm, it readily follows that for a periodically oscillating resource-availability profile, the optimal fill-empty strategy is given by a fill period when resource is available, followed by an empty period when resource is not. This cycling behavior is analogous to the wake-sleep cycles in organismal life, where the generally nocturnal sleep phase is a period where the information collected from the day's activities may be processed. The sleep cycle is then a mechanism for the organism to process a maximal amount of information over a daily cycle. Our model can exhibit phenomena analogous to "microsleeps," and other behavior associated with breakdown in sleep patterns.

Keywords: Sleep, separation of tasks, information processing, optimization

I. INTRODUCTION

Although sleep is exhibited by almost all complex multicellular life, it is still a largely poorly understood phenomenon. The difficulty in understanding the necessity of sleep derives from the observation that sleep-deprived organisms do not show signs of physical damage, and yet they are characterized by impaired cognitive abilities. In extreme cases, sleep deprivation can even lead to death.

A hypothesis for sleep, advanced by Francis Crick, is that sleep is a way for the brain to perform various upkeep, or alternatively, "garbage collecting" functions necessary for proper brain function [1]. More specifically, sleep is a time when the brain sorts through various stored memories, and discards those deemed unessential, while processing those deemed essential for long-term storage.

There is now evidence suggesting that Crick's hypothesis may be correct: It has been discovered that neurons contain a protein, termed Fos, which is involved in proper neuronal function [1]. During periods of neuronal stimulation, Fos naturally builds up, apparently as a by-product of various neuronal activities. Proper neuronal function is no longer possible once Fos levels reach a critical level. During the sleep state of an organism, Fos levels rapidly drop. Apparently, the Fos protein acts as a molecular switch that regulates various genes involved in proper neuronal function [1]. During sleep, these genes are suppressed, allowing the neuron to re-set for the next period of wakefulness.

A number of appropriate analogies for understanding

the phenomenon of sleep include the build-up of lactic acid during periods of intense muscular activity, and the necessity for corporations, or even individuals, to periodically significantly reduce their level of external interactions and to update various inventories and finances.

Based on Crick's hypothesis, we argue that the phenomenon of sleep occurs because, given alternating cycles of available sensory input (day-night), a maximal amount of information processing occurs when the brain devotes its resources to collecting information when it is available, and to then process that information when external information is significantly less available.

In this paper, we proceed to develop a model for the filling and emptying of a tank. Filling the tank corresponds to the input of external information into a biological system (a brain or even a single neuron), while emptying the tank corresponds to the processing of the information. For a given resource availability profile over some time period, we develop a canonical fill-empty profile that we rigorously prove optimizes the total amount of resource processed. We then go on to consider specific resource availability profiles and associated optimal solutions, and draw analogies between the optimal profiles and observed behavior.

We point out that while mathematical models related to sleep have been previously developed [2, 3, 4, 5], such models have primarily focused on the dynamics associated with the underlying pathways controlling the wake-sleep cycle. However, the need for such pathways have not been addressed.

This paper presents a model in which an alternating pattern of resource input and processing emerges as the solution to an optimization problem, which in the case of sleep presumably leads to a fitness advantage for the organism. To our knowledge, a mathematical model whereby sleep emerges as the solution to an optimization

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problem has not been previously advanced.

II. SEPARATION OF TASKS AND OPTIMIZATION OF OUTPUT

Before presenting the actual tank filling model in the following section, we give an example illustrating how separation of tasks can result in a higher level of information processing than if various tasks are performed simultaneously.

So, consider a system that is capable of engaging in two tasks: (1) An “input” task, whereby external resource is input into the system. (2) A “processing” task, whereby the resource is processed for use by the system.

If we are dealing with a biological system, then we assume that the function of each task is mediated by some protein. For the “input” tasks, we denote the protein by P_1 , while for the “processing” tasks, we denote the relevant protein by P_2 . Now, P_1 and P_2 are encoded in a genome, with corresponding genes denoted by G_1 and G_2 . If we assume a total transcription plus translation rate of k_T , then when only one gene is active, the active gene G_i produces protein at a rate k_T , while when both genes are active, each protein is produced at a rate $(1/2)k_T$. We assume that at least one of the genes is active at any given time.

Furthermore, we assume that the proteins have a decay rate given by a first-order constant k_d . Therefore, if $\epsilon_1(t)$ is defined to be 0 if gene G_1 is off, and 1 if gene G_1 is on, and $\epsilon_2(t)$ is defined analogously for gene G_2 , then, letting n_{P_1} , n_{P_2} denote the number of proteins in the system at a given time, we have,

$$\begin{aligned}\frac{dn_{P_1}}{dt} &= \frac{1}{2}(\epsilon_1(t) - \epsilon_2(t) + 1)k_T - k_d n_{P_1} \\ \frac{dn_{P_2}}{dt} &= \frac{1}{2}(\epsilon_2(t) - \epsilon_1(t) + 1)k_T - k_d n_{P_2}\end{aligned}\quad (1)$$

Now, if we let $\delta(t)$ denote a resource availability profile, defined to be 1 when external resource is available, and 0 otherwise, then in the simplest assumption the rate at which resource enters the system is proportional to n_{P_1} , and the rate at which resource is processed is proportional to n_{P_2} . If n_1 denotes the amount of unprocessed resource at any given time t , and n_2 denotes the amount of processed resource, then we have,

$$\begin{aligned}\frac{dn_1}{dt} &= r_1 n_{P_1} \delta(t) - r_2 n_{P_2} (1 - \delta_{n_1,0}) \\ \frac{dn_2}{dt} &= r_2 n_{P_2} (1 - \delta_{n_1,0})\end{aligned}\quad (2)$$

where $\delta_{n_1,0} = 1$ if $n_1 = 0$, and 0 otherwise.

Note then that if $\delta(t)$ experiences oscillatory periods of resource availability, then keeping both genes on at all times may not lead to an optimal processing of resource. The reason for this is that when $\delta(t) = 1$, resource only enters the system at half the maximal rate. Depending

on the resource availability profile, it may be optimal for the system to only take in external resource when it is available, and then process that resource during periods when resource is significantly less available.

We note that for finite values of k_T and k_d , switching from one task to another involves a transient during which the proteins involved in one task degrade and the proteins for the other task reach their steady-state levels. However, if $k_T, k_d \rightarrow \infty$ in such a way that k_T/k_d is fixed, then there is no time associated with switching tasks. In this case, we may assume, for simplicity, that only one task can be active at any given time, since a profile where both tasks are on over a time interval can be approximated to arbitrary accuracy by a profile that rapidly oscillates between one task and the other.

III. A TANK FILLING MODEL

A. Model description

Consider a tank that can be filled and emptied with some unspecified material. At any given time t , external resource is available, or it is not. We denote the resource availability profile by a function $\delta(t)$, where $\delta(t) = 1$ if resource is available at time t , and 0 if not. If $\delta(t) = 1$, then the tank may be filled at a rate r_f . The tank may also be emptied at a rate r_e as long as it is not empty. Over any finite interval $[t_1, t_2]$, we assume that $\delta(t)$ is discontinuous at a finite number of points. This implies that $\delta(t)$ may be taken to be a piecewise constant function.

At any given time t , we assume that the tank is carrying out one of the fill or empty tasks, but that the tasks cannot occur simultaneously. We let $\epsilon_f(t)$ denote the fill profile function, so that $\epsilon_f(t) = 1$ if the tank is in the fill mode at time t , and $\epsilon_f(t) = 0$ otherwise. We let $\epsilon_e(t)$ denote the empty profile function, so that $\epsilon_f(t) + \epsilon_e(t) = 1$ at all times. As with $\delta(t)$, we assume $\epsilon_f(t)$, $\epsilon_e(t)$ are piecewise constant, with a finite number of discontinuities over a finite interval.

We also let $n_T(t)$ denote the total amount of material in the tank at time t , and $n_P(t)$ denote the total amount of material that has been processed through the tank at time t . It should be apparent that,

$$\begin{aligned}\frac{dn_T}{dt} &= r_f \epsilon_f(t) \delta(t) - r_e \epsilon_e(t) (1 - \delta_{n_T,0}) \\ \frac{dn_P}{dt} &= r_e \epsilon_e(t) (1 - \delta_{n_T,0})\end{aligned}\quad (3)$$

B. Optimal fill-empty profiles

Given a resource availability profile $\delta(t)$ over some time interval $[0, T]$, we wish to determine the fill-empty profile $\epsilon = (\epsilon_f, \epsilon_e)$ that maximizes $n_P(T)$, given the initial conditions $n_T(0) = n_P(0) = 0$. As a notational convenience,

we let $n_{\epsilon,T}(t)$, $n_{\epsilon,P}(t)$ denote the n_T and n_P values associated with the fill-empty profile ϵ .

A natural fill-empty profile, denoted $\epsilon_0 = (\epsilon_{f,0}, \epsilon_{e,0})$, is defined by the following prescription: Fill whenever $\delta(t) = 1$, empty whenever $\delta(t) = 0$ as long as $n_T(t) > 0$. Continue with this fill-empty profile until $n_T(t) = r_e(T - t)$, at which point the tank should be emptied until time T . Let t_{ϵ_0} denote the critical time at which emptying until time T begins. Then for notational convenience, we define $I_1 = [0, t_{\epsilon_0}]$, $I_2 = [t_{\epsilon_0}, T]$.

We now prove that ϵ_0 yields a maximal value for $n_P(T)$. We begin by defining, for an arbitrary fill-empty profile ϵ over some set \mathbf{S} , the quantities $T_f(\mathbf{S}; \epsilon)$, $T_e(\mathbf{S}; \epsilon)$, $T_w(\mathbf{S}; \epsilon)$, as follows: We define $\mathbf{S}_f = \{t \in \mathbf{S} | \delta(t) = 1 \text{ and } \epsilon_f(t) = 1\}$, $\mathbf{S}_e = \{t \in \mathbf{S} | n_{\epsilon,T}(t) > 0 \text{ and } \epsilon_e(t) = 1\}$, and $\mathbf{S}_w = \mathbf{S} / (\mathbf{S}_f \cup \mathbf{S}_e)$. If $\mu(\Omega)$ denotes the measure of a set Ω (essentially the total length of the set), then $T_f(\mathbf{S}; \epsilon) \equiv \mu(\mathbf{S}_f)$, $T_e(\mathbf{S}; \epsilon) \equiv \mu(\mathbf{S}_e)$, and $T_w(\mathbf{S}; \epsilon) \equiv \mu(\mathbf{S}_w)$. We should point out that because δ , ϵ_f , ϵ_e are assumed to be piecewise constant, all sets considered in this paper are unions of disjoint intervals, and hence are measurable.

Given a set \mathbf{S} , define $\mathbf{S}^0 = \{t \in \mathbf{S} | \delta(t) = 0\}$, and $\mathbf{S}^1 = \{t \in \mathbf{S} : \delta(t) = 1\}$. Then define $T_{e,0}(\mathbf{S}; \epsilon) = \mu(\mathbf{S}_e^0)$, and $T_{e,1}(\mathbf{S}; \epsilon) = \mu(\mathbf{S}_e^1)$. Note that $T_e(\mathbf{S}; \epsilon) = T_{e,0}(\mathbf{S}; \epsilon) + T_{e,1}(\mathbf{S}; \epsilon)$.

Note that since $\epsilon_{0,f}(t) = 1$ whenever $\delta(t) = 1$ for $t \in I_1$, it follows that $T_f(I_1; \epsilon) \leq T_f(I_1; \epsilon_0) - T_{e,1}(I_1; \epsilon)$.

For any fill-empty profile ϵ , we have,

$$\begin{aligned} n_{\epsilon,T}(T) &= n_{\epsilon,T}(t_{\epsilon_0}) + r_f T_f(I_2; \epsilon) - r_e T_e(I_2; \epsilon) \\ &= r_f T_f(I_1; \epsilon) - r_e T_e(I_1; \epsilon) \\ &\quad + r_f T_f(I_2; \epsilon) - r_e T_e(I_2; \epsilon) \end{aligned} \quad (4)$$

Since $T_f(I_2; \epsilon) + T_e(I_2; \epsilon) \leq T - t_{\epsilon_0}$, we have,

$$\begin{aligned} 0 &\leq n_{\epsilon,T}(T) \\ &\leq r_f (T_f(I_1; \epsilon_0) - T_{e,1}(I_1; \epsilon)) - r_e T_e(I_1; \epsilon) \\ &\quad + r_f (T - t_{\epsilon_0} - T_e(I_2; \epsilon)) - r_e T_e(I_2; \epsilon) \end{aligned} \quad (5)$$

which may be re-arranged to give,

$$\begin{aligned} T_e(I_2; \epsilon) &\leq \frac{r_f T_f(I_1; \epsilon_0) - r_e T_e(I_1; \epsilon)}{r_e} \\ &\quad - \frac{r_f}{r_f + r_e} (T_e(I_1; \epsilon_0) + T_{e,1}(I_1; \epsilon) - T_e(I_1; \epsilon)) \end{aligned} \quad (6)$$

where the derivation of this inequality makes use of the identity $r_f T_f(I_1; \epsilon_0) - r_e T_e(I_1; \epsilon_0) = r_e (T - t_{\epsilon_0})$.

We then obtain that,

$$\begin{aligned} n_{\epsilon,P}(T) &= r_e (T_e(I_1; \epsilon) + T_e(I_2; \epsilon)) \\ &\leq r_f T_f(I_1; \epsilon_0) \\ &\quad - \frac{r_f r_e}{r_f + r_e} (T_e(I_1; \epsilon_0) + T_{e,1}(I_1; \epsilon) \\ &\quad - T_e(I_1; \epsilon)) \end{aligned} \quad (7)$$

Since $n_{\epsilon_0,P}(T) = r_f T_f(I_1; \epsilon_0)$, then if we can show that $T_e(I_1; \epsilon_0) + T_{e,1}(I_1; \epsilon) - T_e(I_1; \epsilon) \geq 0$, we will have established that $n_{\epsilon,P}(T) \leq n_{\epsilon_0,P}(T)$, thereby proving the maximality of ϵ_0 .

So, suppose $T_e(I_1; \epsilon_0) + T_{e,1}(I_1; \epsilon) - T_e(I_1; \epsilon) < 0$. Then defining $I(t) = [0, t]$, we may note that the function $\tau(t) \equiv T_e(I(t); \epsilon_0) + T_{e,1}(I(t); \epsilon) - T_e(I(t); \epsilon)$ is continuous, and satisfies $\tau(0) = 0$, $\tau(t_{\epsilon_0}) < 0$. Let us then define $t^* = \inf\{t \in [0, t_{\epsilon_0}] | \tau(t) < 0\}$. By continuity of τ and from the definition of \inf , we have that $t^* < t_{\epsilon_0}$, $\tau(t^*) = 0$, and that for any $t > t^*$ there exists a $t' \in (t^*, t)$ such that $\tau(t') < 0$.

Now, by assumption $\delta(t)$ is piecewise constant, hence if δ is discontinuous at t^* , then there exists an interval $(t^*, t^* + h) \subset I_1$ over which δ is constant. If δ is continuous at t^* , then there also exists an interval $(t^*, t^* + h) \subset I_1$ over which δ is constant.

Suppose $\delta(t) = 1$ on $(t^*, t^* + h)$. Then for any $h' < h$, we have that $T_e(I(t^* + h'); \epsilon_0) = T_e(I(t^*); \epsilon_0)$, since the prescription for ϵ_0 is to fill when $\delta(t) = 1$ on I_1 . We also have that $T_{e,1}(I(t^* + h'); \epsilon) - T_e(I(t^* + h'); \epsilon) = T_{e,1}(I(t^*); \epsilon) - T_e(I(t^*); \epsilon) + T_{e,1}([t^*, t^* + h']; \epsilon) - T_e([t^*, t^* + h']; \epsilon)$. Since $\delta(t) = 1$ on $(t^*, t^* + h')$, it follows that $T_{e,1}([t^*, t^* + h']; \epsilon) = T_e([t^*, t^* + h']; \epsilon)$, and hence that $T_{e,1}(I(t^* + h'); \epsilon) - T_e(I(t^* + h'); \epsilon) = T_{e,1}(I(t^*); \epsilon) - T_e(I(t^*); \epsilon)$. Therefore, $\tau(t^* + h') = \tau(t^*)$, so that $\tau(t) = 0$ on $[t^*, t^* + h]$, contradicting the definition of t^* .

So, suppose $\delta(t) = 0$ on $(t^*, t^* + h)$. Then it should be clear that $T_{e,1}(I(t); \epsilon)$ is constant over $(t^*, t^* + h)$. If $n_{\epsilon_0,T}(t^*) > 0$, then according to our prescription there exists an $h' \in (0, h)$ such that $\epsilon_e(t) = 0$ with $n_{\epsilon_0,T}(t) > 0$ over $(t^*, t^* + h')$. Therefore, given $h'' \in (0, h')$, we have $T_e(I(t^* + h''); \epsilon_0) = T_e(I(t^*); \epsilon_0) + h''$, while $T_e(I(t^* + h''); \epsilon) = T_e(I(t^*); \epsilon) + T_e([t^*, t^* + h'']; \epsilon)$. The result is that $\tau(t^* + h'') = \tau(t^*) + h'' - T_e([t^*, t^* + h'']; \epsilon) \geq \tau(t^*) = 0$. Therefore, $\tau(t) \geq 0$ on $[t^*, t^* + h]$, again contradicting the definition of t^* .

So, suppose that $\delta(t) = 0$ on $(t^*, t^* + h)$ with $n_{\epsilon_0,T}(t^*) = 0$. Then,

$$\begin{aligned} n_{\epsilon,T}(t^*) &= r_f T_f(I(t^*); \epsilon) - r_e T_e(I(t^*); \epsilon) \\ &\leq r_f (T_f(I(t^*); \epsilon_0) - T_{e,1}(I(t^*); \epsilon)) \\ &\quad - r_e (T_e(I(t^*); \epsilon_0) + T_{e,1}(I(t^*); \epsilon)) \\ &= n_{\epsilon_0,T}(t^*) - (r_f + r_e) T_{e,1}(I(t^*); \epsilon) \\ &= -(r_f + r_e) T_{e,1}(I(t^*); \epsilon) \end{aligned} \quad (8)$$

which is only possible if $n_{\epsilon,T}(t^*) = 0$ with $T_{e,1}(I(t^*); \epsilon) = 0$. But, since $n_{\epsilon,T}(t^*) = n_{\epsilon_0,T}(t^*) = 0$, then since $\delta(t) = 0$ on $(t^*, t^* + h)$, it follows that $n_{\epsilon,T}(t) = n_{\epsilon_0,T}(t) = 0$ on $(t^*, t^* + h)$, and hence $T_{e,1}([t^*, t^* + h']; \epsilon) = T_e([t^*, t^* + h']; \epsilon) = T_e([t^*, t^* + h']; \epsilon_0) = 0$ on for all $h' \in [0, h]$, so that $\tau(t) = \tau(t^*) = 0$ on $[t, t + h]$, which is again a contradiction.

Since we have exhausted all possibilities, we have established that $\tau(t_{\epsilon_0}) < 0$ leads to a contradiction. Therefore, $\tau(t_{\epsilon_0}) \geq 0$, and the proof is complete.

We should note that, although the optimal fill-empty profile ϵ_0 may not necessarily be unique, if ϵ denotes any other optimal fill-empty profile, then we must have $T_e([0, T]; \epsilon) = T_e([0, T]; \epsilon_0)$. It may be readily shown that $n_{\epsilon, T}(T) = n_{\epsilon_0, T}(T) = 0$:

An ϵ for which $n_{\epsilon, T}(T) > 0$ is not optimal, for letting t_ϵ denote when $n_{\epsilon, T}(t) = r_e(T - t)$ (by the Intermediate Value Theorem, such a t exists), we have $n_{\epsilon, P}(T) = n_{\epsilon, P}(t_\epsilon) + r_e T_e([t_\epsilon, T]; \epsilon) \leq n_{\epsilon, P}(t_\epsilon) + r_e(T - t_\epsilon)$, with equality only occurring when $T_e([t_\epsilon, T]; \epsilon) = r_e(T - t_\epsilon)$. However, $T_e([t_\epsilon, T]; \epsilon) = r_e(T - t_\epsilon)$ implies that $n_{\epsilon, T}(T) = 0$, and so our claim is proved.

But this implies that $r_f T_f([0, T]; \epsilon) = r_e T_e([0, T]; \epsilon) = r_e T_e([0, T]; \epsilon_0) = r_f T_f([0, T]; \epsilon_0)$, so that $T_f([0, T]; \epsilon) = T_f([0, T]; \epsilon_0)$. Finally, $T_w([0, T]; \epsilon) = T - T_f([0, T]; \epsilon) - T_e([0, T]; \epsilon) = T - T_f([0, T]; \epsilon_0) - T_e([0, T]; \epsilon_0) = T_w([0, T]; \epsilon_0)$.

Therefore, although the optimal fill-empty profile may not be unique, the T_f , T_e , and T_w values are uniquely specified.

C. Examples of optimal fill-empty profiles

For simplicity, we consider optimal fill-empty profiles generated by a $\delta(t)$ that is periodic over the time interval $[0, T]$. Specifically, we consider a basic profile, denoted $\sigma_{(T_1, T_2)}$, defined by,

$$\sigma_{(T_1, T_2)}(t) = \begin{cases} 1 & \text{if } t \in [0, T_1] \\ 0 & \text{if } t \in (T_1, T_1 + T_2] \end{cases}$$

Over the time interval $[0, T]$, we then define $\delta(t)$ by setting $\delta(t) = \sigma_{(T_1, T_2)}(t)$ on $[0, T_1 + T_2]$, and then imposing the periodicity relation $\delta(t) = \delta(t + T_1 + T_2)$. We also assume that $T = N(T_1 + T_2)$, for some positive integer N .

Now, if $r_e T_2 \geq r_f T_1$, then it should be apparent that an optimal fill-empty profile, defined by our ϵ_0 prescription, is to fill whenever $\delta(t) = 1$, and to empty whenever $\delta(t) = 0$ and $n_{\epsilon, T}(t) > 0$, until time T . For this profile, then, we have that the optimal values of T_f , T_e , and T_w are given by,

$$\begin{aligned} T_f([0, T]; \epsilon_0) &= NT_1 \\ T_e([0, T]; \epsilon_0) &= \frac{r_f}{r_e} NT_1 \\ T_w([0, T]; \epsilon_0) &= N \frac{r_e T_2 - r_f T_1}{r_e} \end{aligned} \quad (9)$$

and so any other optimal fill-empty profile must fill exactly when $\delta(t) = 1$. Note that T_w only occurs in intervals where $\delta(t) = 0$ (since $T_f([0, T]; \epsilon_0) = NT_1$). Therefore, during these periods, since there is a lack of available resource (analogous to periods of night when making the analogy to sleep), the tank is either filling when there is nothing to fill it with (analogous to wakefulness periods during lack of available sensory information), or is emptying when the tank is already empty (analogous to excess

sleeping). When $\delta(t) = 0$, any combination of filling, or emptying when $n_T = 0$, over these time intervals will not affect the final value of $n_P(T)$. Thus, there is a considerable degree of freedom in choosing an optimal fill-empty profile, which is consistent with the observed breakdown in sleeping patterns in environments with limited exposure to the sun.

Now consider the case when $r_e T_2 < r_f T_1$. Then at the end of every resource availability cycle $[n(T_1 + T_2), (n + 1)(T_1 + T_2)]$, we have that $n_{\epsilon_0, T}$ increases by $r_f T_1 - r_e T_2$, as long as we are before the final emptying phase. So, if n is chosen so that $n(r_f T_1 - r_e T_2) \leq r_e(T - n(T_1 + T_2))$ but $(n + 1)(r_f T_1 - r_e T_2) > r_e(T - (n + 1)(T_1 + T_2))$, then we have $t_{\epsilon_0} \in [n(T_1 + T_2), (n + 1)(T_1 + T_2)]$, which further implies that $t_{\epsilon_0} \in [n(T_1 + T_2), n(T_1 + T_2) + T_1]$. Therefore, $n_{\epsilon, T}(t_{\epsilon_0}) = n(r_f T_1 - r_e T_2) + r_f(t_{\epsilon_0} - n(T_1 + T_2)) = r_e(T - t_{\epsilon_0})$.

Solving for t_{ϵ_0} , we obtain,

$$t_{\epsilon_0} = nT_2 + \frac{r_e}{r_e + r_f} T \quad (10)$$

from which it follows that,

$$\begin{aligned} T_f([0, T]; \epsilon_0) &= \frac{r_e}{r_e + r_f} T \\ T_e([0, T]; \epsilon_0) &= \frac{r_f}{r_e + r_f} T \\ T_w([0, T]; \epsilon_0) &= 0 \end{aligned} \quad (11)$$

An interesting fill-empty profile that arises from this condition is defined as follows: Over the interval $[0, T]$, $\epsilon_e(t) = 1$ whenever $\delta(t) = 0$. Whenever $\delta(t) = 1$, $\epsilon_f(t)$ and $\epsilon_e(t)$ alternate in being 1, with corresponding time lengths t_f , t_e . To determine these time lengths, we first assume that there are M such cycles over each $\delta(t) = 1$ interval of length T_1 , so that $t_f + t_e = T_1/M$. The net accumulation in the tank over each such interval should be $r_e T_2 = M(r_f t_f - r_e t_e)$, which may be solved to give,

$$\begin{aligned} t_f &= \frac{1}{M} \frac{r_e}{r_f + r_e} (T_1 + T_2) \\ t_e &= \frac{1}{M} \frac{r_f T_1 - r_e T_2}{r_f + r_e} \end{aligned} \quad (12)$$

Note that $r_f t_f > r_e t_e$, and that $T_e([0, T]; \epsilon) = r_f/(r_e + r_f) T$, so that this profile is indeed an optimal one.

Essentially, when the amount of time during which $\delta(t) = 0$ is not sufficiently long to process all of the resource that can fill the tank, then one optimal solution profile is, during periods when $\delta(t) = 1$, to fill and empty the tank in alternating time intervals of length t_f , t_e , and then to empty the tank whenever $\delta(t) = 0$. We argue that the t_e empty periods are analogous to the phenomenon of ‘‘microsleep’’ that occurs during sleep deprivation.

In general, we claim that any optimal solution profile for this form of $\delta(t)$ will have $\epsilon_e(t) = 1$ whenever $\delta(t) = 0$. Otherwise, we obtain, $r_f/(r_f + r_e) T = T_e([0, T]; \epsilon) = T_{e,0}([0, T]; \epsilon) + T_{e,1}([0, T]; \epsilon) < NT_2 + T_{e,1}([0, T]; \epsilon) \Rightarrow T_{e,1}([0, T]; \epsilon) > N(r_f T_1 - r_e T_2)/(r_f + r_e)$.

But this implies that $T_f([0, T]; \epsilon) \leq NT_1 - T_{e,1}([0, T]; \epsilon) < NT_1 - N(r_f T_1 - r_e T_2)/(r_f + r_e) = r_e/(r_e + r_f)T \Rightarrow \Leftarrow$. With this contradiction, our claim is

proved.

As a final example for this subsection, we consider a resource availability profile given by,

$$\delta(t) = \begin{cases} \sigma_{(T_1, T_2)}(t) & \text{if } t \in [0, T_1 + T_2] \\ \sigma_{(T_1, T'_2)}(t + T_1 + T_2) & \text{if } t \in [T_1 + T_2, T = 2T_1 + T_2 + T'_2] \end{cases}$$

where $r_e T'_2 < r_f T_1 < r_e T_2$. Then it is possible to show that,

$$\begin{aligned} T_f([0, T]; \epsilon_0) &= T_1 + \frac{r_e}{r_e + r_f}(T_1 + T'_2) \\ T_e([0, T]; \epsilon_0) &= \frac{r_f}{r_e} T_1 + \frac{r_f}{r_e + r_f}(T_1 + T'_2) \\ T_w([0, T]; \epsilon_0) &= \frac{r_e T_2 - r_f T_1}{r_e} \end{aligned} \quad (13)$$

We now show that this uniquely determines $T_f([0, T_1 + T_2]; \epsilon_0)$, $T_f([T_1 + T_2, T]; \epsilon_0)$, $T_e([0, T_1 + T_2]; \epsilon_0)$, $T_e([T_1 + T_2, T]; \epsilon_0)$, $T_w([0, T_1 + T_2]; \epsilon_0)$, $T_w([T_1 + T_2, T]; \epsilon_0)$.

Note that $T_f([0, T]; \epsilon_0) \geq T_f([0, T_1 + T_2]; \epsilon_0) + \frac{r_e}{r_e + r_f}(T_1 + T'_2) \Rightarrow T_f([T_1 + T_2, T]; \epsilon_0) \geq \frac{r_e}{r_e + r_f}(T_1 + T'_2)$. Therefore, $T_e([T_1 + T_2, T]; \epsilon_0) \leq \frac{r_f}{r_e + r_f}(T_1 + T'_2)$. However, $n_{\epsilon_0, T}(T) = 0 \geq r_f T_f([T_1 + T_2, T]; \epsilon_0) - r_e T_e([T_1 + T_2, T]; \epsilon_0) \geq \frac{r_f r_e}{r_e + r_f}(T_1 + T'_2) - r_e T_e([T_1 + T_2, T]; \epsilon_0) \Rightarrow T_e([T_1 + T_2, T]; \epsilon_0) \geq \frac{r_f}{r_e + r_f}(T_1 + T'_2)$, and so $T_e([T_1 + T_2, T]; \epsilon_0) = \frac{r_f}{r_e + r_f}(T_1 + T'_2)$. But this implies that $T_f([T_1 + T_2, T]; \epsilon_0) \leq \frac{r_e}{r_e + r_f}(T_1 + T'_2)$, so $T_f([T_1 + T_2, T]; \epsilon_0) = \frac{r_e}{r_e + r_f}(T_1 + T'_2)$. This further implies that $T_w([T_1 + T_2, T]; \epsilon_0) = 0$, and so that $T_w([0, T_1 + T_2]; \epsilon_0) = \frac{r_e T_2 - r_f T_1}{r_e}$. Finally, we obtain $T_f([0, T_1 + T_2]; \epsilon_0) = T_1$, and $T_e([0, T_1 + T_2]; \epsilon_0) = \frac{r_f}{r_e} T_1$.

Summarizing the results, we have,

$$\begin{aligned} T_f([0, T_1 + T_2]; \epsilon_0) &= T_1 \\ T_f([T_1 + T_2, T]; \epsilon_0) &= \frac{r_e}{r_e + r_f}(T_1 + T'_2) < T_1 \\ T_e([0, T_1 + T_2]; \epsilon_0) &= \frac{r_f}{r_e} T_1 \\ T_e([T_1 + T_2, T]; \epsilon_0) &= \frac{r_f}{r_f + r_e}(T_1 + T'_2) > T'_2 \\ T_w([0, T_1 + T_2]; \epsilon_0) &= \frac{r_e T_2 - r_f T_1}{r_e} \\ T_w([T_1 + T_2, T]; \epsilon_0) &= 0 \end{aligned} \quad (14)$$

Therefore, maximal processing of external resource still requires emptying the tank when $\delta(t) = 1$ on the interval $[T_1 + T_2, T]$. In making the analogy with sleep, this implies that excess sleep in a certain time interval will not prevent sleep deprivation in a later time interval when the period of resource availability exceeds the

period when the resource may be processed. Intuitively, this makes sense, since, once the brain has “taken out the trash,” further sleep will not prevent the accumulation of “trash” during a later cycle.

IV. CONCLUSIONS AND FUTURE RESEARCH

This paper presented a simplified model of the filling and emptying of a tank as an analogy with which to analyze the phenomenon of sleep. Our model is motivated by a hypothesis that sleep is the brain’s way of processing material accumulated during periods of sensory input. By analogy with a corporation, which may need to periodically update inventories and finances for proper functioning, so too the brain may need to periodically cut-off external inputs and process accumulated material, in order to maintain updated information with which to properly analyze the environment.

Based on our tank filling model, we were able to show that when there are alternating periods of availability of external inputs, the optimal amount of information can be processed if information is collected when it is available (during the day for the vast majority of organisms), and processed when it is not, or considerably less, available (at night, again, for the vast majority of organisms). We showed that when there are excessive periods where resource is unavailable, then there is a continuum of optimal fill-empty profiles, involving either filling when no resource is present, or emptying when the tank is empty. Thus, our model is consistent with the observed breakdown in sleeping patterns in environments with minimal exposure to the sun.

Furthermore, we showed that when the periods of information deficiency are not sufficiently long to process all of the collected information, an optimal information collection strategy can involve periodic tank emptying intervals during periods of information availability. We argued that this behavior is analogous to the observed phenomenon of “microsleeps.”

Our model also suggests that it is not possible to store sleep time beyond a certain amount, so that periods of extended sleep will not prevent sleep deprivation at a later time. Our model also captures the breakdown in sleeping patterns

Although our model is consistent with a number of

phenomena associated with sleep, it is nevertheless highly simplistic: Our model assumes that the tank can be filled without limit, corresponding to an organismal ability to maintain unlimited periods of wakefulness. This of course is incorrect, because an over-accumulation of unprocessed information will eventually lead to improper brain function. Therefore, one important modification that must be made to our model is to place an upper bound on n_T , denoted $n_{T,max}$, so that the fill-empty profile must be such that $n_{\epsilon,T}(t) \leq n_{T,max}$ at all times.

Furthermore, our model also assumes that there is no time delay associated with switching tasks (from fill to empty and vice versa). In reality, when task switching occurs, there are generally transients associated with the time it takes for the components for performing one task to shut-down, and for another set of components for performing the other task to start up. These transients lead to a time cost τ_{switch} during which neither system is active, and so future models will need to consider positive values of τ_{switch} when determining optimal fill-empty profiles.

In addition, pursuing the analogy with a corporation, we may note that although inventories and finances may be updated at regular intervals, it may nevertheless be an optimal strategy to also perform some level of upkeep during periods of external input. A lower level of upkeep could have shorter associated transients, and by lessening the rate of accumulation of material, will allow for a greater overall input of external resource before system-wide upkeep becomes necessary.

Our model considered the filling and emptying of a sin-

gle tank. In reality, an organ such as the brain consists of billions of interacting neurons, and thus our model should be extended to include interactions amongst numerous tanks. An important issue to consider is neuronal synchronization, whereby the neurons oscillate between their waking and sleep modes together. Presumably, since it is advantageous for each component of a system to switch between input and processing tasks in accordance with external resource availability, and since system performance is optimized when all components coordinate their actions, the strategy whereby all system components collectively switch between input and processing tasks in accordance with external resource availability is a self-reinforcing one.

As a final remark, because sleep arises as the need for the brain to perform upkeep tasks that accumulate over a period of sensory input, mathematical models related to sleep could also be useful for analyzing other interconnected structures where a proper ordering of tasks could lead to optimal system performance. In particular, models of sleep could be useful in understanding systems that exhibit oscillatory periods of relatively intense apparent activity (as exhibited by absorption of information or material into the system), followed by periods of apparent stasis.

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